



## Coefficient Inequalities for Certain New Class of Analytic Functions Related with q-starlike Exponential Function

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### Abstract

The purpose of the present work is to define a class of q-starlike functions in the domain of exponential function related to q-derivative operator, which is subordinate to exponential function  $\frac{{}^z D_q f(z)}{f(z)} \prec e^z$ . for this function class  $S_{q_l}^*$  associated with exponential function. We will determine the possible upper bound of determinant  $H_3(1)$ .

**Keywords:** Analytic Functio<sup>1</sup>, Hankel Determinant<sup>2</sup>, q-derivative<sup>3</sup>, Symmetric Point<sup>4</sup>.

### 1. Introduction

An analytical function, also called a holomorphic function, is a function that is defined on an open subset of the complex plane and is complex differentiable at all points within that subset in complex analysis and geometry. The class of all analytic function  $f$  with the normalized condition in the open unit disc  $E = \{z: |z| < 1\}$  is a symbol by  $A$  and has the Taylor series we have

$$f(\tilde{z}) = \tilde{z} + \alpha_2 \tilde{z}^2 + \alpha_3 \tilde{z}^3 + \alpha_4 \tilde{z}^4 + \dots \quad (1)$$

The class of univalent & analytic functions unit disk  $E$  is prove by S. Carathéodory function are a category of complex valued function defined on the unit disk of the complex plane in geometric function theory. The theory of conformal mappings is strongly related to these functions. After the Greek mathematician Constantin Carathéodory, who made substantial contributions to geometric function theory, the idea of Carathéodory functions was named.

$$p(\tilde{z}) = 1 + p_1 \tilde{z} + p_2 \tilde{z}^2 + p_3 \tilde{z}^3 + p_4 \tilde{z}^4 + \dots \quad (2)$$

The Schwarz function, named after the German mathematician Hermann Schwarz, is a complex valued function that maps the unit disk  $E$  in the complex planes to itself. It is known by  $f(z) = \frac{-z}{1-z^2}$  where  $z$  is a complex number.



Specifically, if  $f$  and  $g$  are analytic function defined on some domain  $D$ , then  $f < g$  if there lie other analytic function  $h$  defined on  $D$  such that,  $f(z) = g(h(z)) \forall z \text{ in } D$ . see ([2],[3]). Mendiratta et al [4] introduced the exponential function's associated  $S_l^*$  subclass of analytical functions. Pommerenke [1], Noonan & Thomas [5]. Hankel determinant  $H_k(c)$  is defined, for any positive integer  $k$  and corresponding coefficients  $c$ , for functions belonging to the class  $S$ .

$$H_k(c) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{3}$$

For fixed positive integers  $k$  and  $c$ , the growth behavior of the Hankel determinant  $H_k(c)$  as  $n \rightarrow \infty$  was studied by Noor [6] in 1983, who established results concerning its boundedness. Later, Ehrenborg [7] examined the Hankel determinant associated with exponential polynomials. The determinant corresponding to different orders of differentiability depends on the variation of  $k$  and  $c$ . In particular, for the case  $k=2$  and  $c=1$ , it is defined as follows:

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}, \quad |a_1 a_3 - a_2^2|, \quad a_1 = 1 \tag{4}$$

The Fekete-Szego inequality, which is well-known, is a result of complex analysis and potential theory and offers an estimate for the expansion of the analytical unit disk  $E$  Taylor coefficients. More precisely, let  $f(z)$  be functions that are analytical in the open unit disc  $|z| < 1$ , and let its Taylor series expansions be given by (1). In 1933, [8] For the coefficient of a univalent analytical function on the unit disk, Fekete-Szego defined an inequality.

Let  $f$  be a univalent function then  $|a_3 - \eta a_2^2| \leq 1 + 2 \exp(\frac{-2\eta}{1-\eta}), 0 \leq \eta < 1$ . The Fekete-Szego coefficient

functional is a natural outgrowth of the study of the univalence of analytical functions. Many authors have investigated the Fekete-Szego role in various univalent subclasses, see [9]. other researcher like Ali et al [10],[11], Owa and Cho [12],[13], Merkes and koegh [14]. In 2007, Murugusundaramoorthy et al. [15] investigated the Fekete-Szego inequality, recognized in the open unit disc, for a number of normalized analytical functions  $f(z)$ . Now for  $k = 2, c = 2$  it can be obtained that

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \tag{5}$$

$$= |a_2 a_4 - a_3^2|.$$

In 1968, Hayman [16] introduced the concept of the second Hankel determinant in the study of univalent functions. Subsequently, several researchers have contributed to this area, including Janteng et al. [17, 18], Bansal [19], Lee et al. [20], Lei et al. [21], Rain et al. [22], Rajya et al. [23], Zaprawa [24], Shi et al. [25], Ullah et al. [26], and Obradović et al. [27]. Zaprawa [24] further investigated the upper bound of the Hankel determinant for the coefficients of functions  $f$  belonging to the class  $S$  of univalent functions or its subclasses. In particular, for  $k = 2$  and  $c = 3$ , the Hankel determinant  $H_2(3)$  is defined as follows:

$$H_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = |a_3 a_5 - a_4^2|. \tag{6}$$

For  $k = 3, c = 1$  the Hankel Determinant,  $H_3(1)$  is know as 3rd Hankel Deter-minant we have

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, = a_3(a_2 a_4 - a_3^2) - a_4(a_1 a_4 - a_2 a_3) + a_5(a_3 - a_2^2), \quad a_1 = 1 \tag{7}$$

The methodology known as *q-calculus* is analogous to classical calculus; however, it deals with the derivation of q-analogous results without the direct use of limits. The systematic development of q-calculus is attributed to Jackson [37], who also introduced and defined the concept of the *q-derivative* [38]. The *q-derivative* of a function  $f$ , where  $f$  is a normalized analytic function, is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, z \neq E$$

and  $D_q f(0) = f'(0)$ , where  $q \in (0,1)$ . As  $q \rightarrow 1^-$ , the q-derivative  $D_q f$  approaches the classical derivative  $f'$ . In 2007, Babalola [24] was the first to analyze the upper bound of the third Hankel determinant for certain subclasses of the class  $S$  of univalent functions. In 2016, Vamshee Krishna et al [29] introduced toeplitz determinant to give the tops upper bounds of a  $H_3(1)$  Hankel determinants for starlike function with regard to the symmetrical point. In 2017 Prajapat et al [30] investigated the Taylor coefficients of functions  $f(z)$  belongings to a definite class of an analytical function in the open unit disc  $E$  are explored to determine the 3rd Hankel determinant. Other researchers like Yalcin and Altinkaya [31], Cho et al. [32], Mohd Narzan et al. [35], Lecko et al. [33], and Kowalczyk et al [39]. Breaz et al [40], Singh [41], Gurusang et al [42].

**Definition 1.** A functions  $f \in S$  and  $f$  is know to be in the class  $S_{qt}^*(e^z)$  as

$$\frac{zD_q f(z)}{f(z)} \prec e^z, z \in D \tag{8}$$

It is worth noting that, by taking  $q \rightarrow 1^-$  in the above definition, we obtain the well-known class  $S_t^*$  [36]. The following lemmas are presented to establish the desired results.

**Lemma 2** [43] If  $p(z) \in P$ , then exists some  $x, z$  with  $|x| \leq 1, |z| \leq 1$  we have

$$2c_2 = c_1^2 + x(4 - c_1^2). \tag{9}$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{10}$$

**Lemma 3** [44] If  $p \in P$ , then  $|c_n| \leq 2, \forall n \in N$ .

**Lemma 4** [4] If the function  $f(z) \in S_t^*$  and of the equation (1) we have

$$|a_2| = 1, |a_3| = \frac{3}{4}, |a_4| = \frac{17}{36}, |a_5| = 1.$$

## 2. Main Results

For the following results we suppose  $\alpha = [2]_q - [1]_q, \beta = [3]_q - [1]_q, \delta = [4]_q - [1]_q, q \in (0,1..$

**Theorem 5** If  $f \in S_{qt}^*$  then  $|a_2| \leq \frac{1}{\beta}$ .

**Proof:** As  $f \in S_{q1}^*(e^z)$  as

$$\frac{zD_q f(z)}{f(z)} = e^{w(z)} \tag{11}$$

Using (1), we consider

$$\frac{zD_q f(z)}{f(z)} = [1]_q + 2(\alpha)a_2z + ((\beta)a_3 + ([1]_q - [2]_q)a_2^2)z^2 + (\delta)a_4 + ([2]_q - [5]_q)a_3a_2 + [2]_q - [1]_q a_2^3 z^3 + \dots \tag{12}$$

$$p(z) = \frac{1 + w(z)}{1 - w(z)}$$

Equivalent,

$$w(z) = \frac{p(z)-1}{p(z)+1} \tag{13}$$

Consider

$$e^{w(z)} = 1 + w(z) + \frac{(w(z))^2}{2!} + \frac{(w(z))^3}{3!} + \frac{(w(z))^4}{4!} + \dots \tag{15}$$

We get

$$e^{w(z)} = 1 + \frac{c_1 z}{2} + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1 c_2}{4} + \frac{c_1^3}{48}\right)z^3 + \dots \tag{16}$$

From (12) and (16) we compare the coefficient, we get

$$a_2 \leq \frac{c_1}{2(\alpha)}, a_3 \leq \frac{c_2}{2(\beta)} - \frac{c_1^2}{8(\beta)} + \frac{([1]_q - [2]_q)c_1^2}{(2(\alpha))^2(\beta)}, a_4 = \frac{c_3}{2(\delta)} - \frac{c_1 c_2}{4(\delta)} + \frac{([5]_q - [2]_q)c_1 c_2}{4(\beta)(\alpha)(\delta)} + \frac{c_1^3}{48(\delta)} - \frac{c_1^3([5]_q - [2]_q)}{16(\beta)(\alpha)(\delta)} + \frac{c_1^3([5]_q - [2]_q)([1]_q - [2]_q)}{(\beta)(2(\alpha))^2(\delta)2(\alpha)} + \frac{c_1^3([1]_q - [2]_q)}{(\delta)(2(\alpha))^3} \tag{17}$$

As

$$|a_3 - a_2^2| = \left| \frac{c_2}{2(\beta)} - \frac{c_1^2}{8(\beta)} + \frac{([1]_q - [2]_q)c_1^2}{(2(\alpha))^2(\beta)} - \frac{c_1^2}{(2(\alpha))^2} \right|$$

Using Lemma 2 we have

$$|a_3 - a_2^2| = \left| \frac{x(4 - c_1^2)}{4(\beta)} - \frac{c_1^2}{8(\beta)} + \frac{([1]_q - [2]_q)c_1^2}{(2(\alpha))^2(\beta)} - \frac{c_1^2}{(2(\alpha))^2} \right|$$

As  $|x| = t \in [0,1]$ ,  $C_1 = c \in [0, 2]$  and apply the triangle inequality we have,

$$|a_3 - a_2^2| \leq \frac{t(4 - c_1^2)}{4(\beta)} + \frac{c_1^2}{8(\beta)} - \frac{([1]_q - [2]_q)c_1^2}{(2(\alpha))^2(\beta)} + \frac{c_1^2}{(2(\alpha))^2}$$

Suppose that

$$F(c, t) = \frac{t(4 - c_1^2)}{4(\beta)} + \frac{c_1^2}{8(\beta)} - \frac{([1]_q - [2]_q)c_1^2}{(2(\alpha))^2(\beta)} + \frac{c_1^2}{(2(\alpha))^2}$$

We have

$$\frac{\partial F}{\partial t} = \frac{(4 - c_1^2)}{4(\beta)}$$

The functions  $F(c, t)$  can get max value at  $t = 1$ , we get

$$\max F(c, t) = F(c, 1) = \frac{(4 - c_1^2)}{4(\beta)} + \frac{c_1^2}{8(\beta)} - \frac{([1]_q - [2]_q)c_1^2}{(2(\alpha))^2(\beta)} + \frac{c_1^2}{(2(\alpha))^2}.$$

Let

$$G(c) = \frac{(4 - c_1^2)}{4(\beta)} + \frac{c_1^2}{8(\beta)} - \frac{([1]_q - [2]_q)c_1^2}{(2(\alpha))^2(\beta)} + \frac{c_1^2}{(2(\alpha))^2}. \tag{18}$$

Max value at c=0 we have

$$|a_3 - a_2^2| \leq G(0) = \frac{1}{\beta}. \tag{19}$$

This is required result.

**Theorem 6** If  $f \in S_{qt}^*$  then  $|a_2 a_3 - a_4| \leq 0.5351q$ .

**Proof:** From (17) in Theorem 5 we have simplifying, we get

$$|a_2 a_3 - a_4| = \left| \left( \frac{c_1 c_2}{4(\beta)(\alpha)} + \frac{c_1 c_2}{4(\delta)} - \frac{([5]_q - [2]_q)c_1 c_2}{4(\beta)(\alpha)(\delta)} \right) - \frac{c_3}{2(\delta)} + \frac{([1]_q - [2]_q)c_1^3}{2(\alpha)(2(\alpha))^2(\beta)} - \frac{c_1^3}{16(\beta)(\alpha)} - \frac{c_1^3}{48(\delta)} + \frac{([5]_q - [2]_q)c_1^3}{16(\alpha)(\delta)(\beta)} - \frac{([1]_q - [2]_q)([1]_q - [2]_q)c_1^3}{2(\alpha)(2(\alpha))^2(\beta)(\delta)} - \frac{([1]_q - 2q)c_1^3}{(\delta)(2(\alpha))^3} \right|. \tag{20}$$

By using Lemma 2, we get

$$|a_2 a_3 - a_4| = \left| \left( \frac{(4 - c_1^2)c_1 x^2}{8(\delta)} + \frac{(4 - c_1^2)c_1 x}{8(\beta)(\alpha)} + \frac{(4 - c_1^2)c_1 x}{8(\delta)} + \frac{(4 - c_1^2)c_1 x([5]_q - [2]_q)}{8(\beta)(\alpha)(\delta)} - \frac{(4 - c_1^2)c_1 x}{4(\delta)} - \frac{(4 - c_1^2)(1 - |x|^2)z}{4(\delta)} + \frac{c_1^3}{8(\beta)(\alpha)} + \frac{c_1^3}{8(\delta)} - \frac{(\beta)c_1^3}{8(\alpha)(\beta)(\delta)} - \frac{c_1^3}{8(\delta)} - \frac{c_3}{2(\delta)} + \frac{([1]_q - [2]_q)c_1^3}{2(\alpha)(2(\alpha))^2(\beta)} - \frac{c_1^3}{16(\beta)(\alpha)} - \frac{c_1^3}{48(\delta)} + \frac{([5]_q - [2]_q)c_1^3}{16(\alpha)(\delta)(\beta)} - \frac{([1]_q - [2]_q)([1]_q - [2]_q)c_1^3}{2(\alpha)(2(\alpha))^2(\beta)(\delta)} - \frac{([1]_q - 2q)c_1^3}{(\delta)(2(\alpha))^3} \right|. \tag{21}$$

Assume that  $|x| = t \in [0, 1], c_1 = c \in [0, 2]$ . using the triangle inequality we have

$$|a_2 a_3 - a_4| \leq \frac{(4 - c^2)ct^2}{8(\delta)} + \frac{(4 - c^2)ct}{8(\beta)(\alpha)} + \frac{(4 - c^2)ct}{8(\delta)} + \frac{(4 - c^2)ct([5]_q - [2]_q)}{8(\beta)(\alpha)(\delta)} - \frac{(4 - c^2)ct}{4(\delta)} + \frac{(4 - c^2)}{4(\delta)} + \frac{c^3}{8(\beta)(\alpha)} + \frac{c^3}{8(\delta)} - \frac{(\beta)c^3}{8(\alpha)(\beta)(\delta)} - \frac{c^3}{8(\delta)} - \frac{c_3}{2(\delta)} + \frac{([1]_q - [2]_q)c^3}{2(\alpha)(2(\alpha))^2(\beta)} - \frac{c^3}{16(\beta)(\alpha)} - \frac{c^3}{48(\delta)} + \frac{([5]_q - [2]_q)c^3}{16(\alpha)(\delta)(\beta)} - \frac{([1]_q - [2]_q)c^3}{2(\alpha)(2(\alpha))^2(\beta)(\delta)} - \frac{([1]_q - 2q)c^3}{(\delta)(2(\alpha))^3}.$$

This gives us

$$F(c, t) = \frac{(4 - c^2)ct^2}{8(\delta)} + \frac{(4 - c^2)ct}{8(\beta)(\alpha)} + \frac{(4 - c^2)ct}{8(\delta)} + \frac{(4 - c^2)ct([5]_q - [2]_q)}{8(\beta)(\alpha)(\delta)} - \frac{(4 - c^2)ct}{4(\delta)} + \frac{(4 - c^2)}{4(\delta)} + \frac{c^3}{8(\beta)(\alpha)} + \frac{c^3}{8(\delta)} - \frac{(\beta)c^3}{8(\alpha)(\beta)(\delta)} - \frac{c^3}{8(\delta)} - \frac{c_3}{2(\delta)} + \frac{([1]_q - [2]_q)c^3}{2(\alpha)(2(\alpha))^2(\beta)} - \frac{c^3}{16(\beta)(\alpha)} - \frac{c^3}{48(\delta)} + \frac{([5]_q - [2]_q)c^3}{16(\alpha)(\delta)(\beta)} - \frac{([1]_q - [2]_q)c^3}{2(\alpha)(2(\alpha))^2(\beta)(\delta)} - \frac{([1]_q - 2q)c^3}{(\delta)(2(\alpha))^3}.$$

$$\frac{\partial F}{\partial t} = \frac{(4 - c^2)ct}{4(\delta)} + \frac{(4 - c^2)c}{8(\beta)(\alpha)} + \frac{(4 - c^2)c}{8(\delta)} + \frac{(4 - c^2)c([5]_q - [2]_q)}{8(\beta)(\alpha)(\delta)} - \frac{(4 - c^2)c}{4(\delta)}.$$

This implies that the max value of  $F(c, t)$  as  $t = 1$ , we get

$$\max F(c, t) = F(c, 1) = \frac{(4 - c^2)c}{8(\delta)} + \frac{(4 - c^2)c}{8(\beta)(\alpha)} + \frac{(4 - c^2)c}{8(\delta)} + \frac{(4 - c^2)c([5]_q - [2]_q)}{8(\beta)(\alpha)(\delta)} - \frac{(4 - c^2)c}{4(\delta)} + \frac{(4 - c^2)}{4(\delta)} + \frac{c^3}{8(\beta)(\alpha)} + \frac{c^3}{8(\delta)} - \frac{(\beta)c^3}{8(\alpha)(\beta)(\delta)} - \frac{c^3}{8(\delta)} - \frac{c_3}{2(\delta)} + \frac{([1]_q - [2]_q)c^3}{2(\alpha)(2(\alpha))^2(\beta)} - \frac{c^3}{16(\beta)(\alpha)} - \frac{c^3}{48(\delta)} + \frac{([5]_q - [2]_q)c^3}{16(\alpha)(\delta)(\beta)} - \frac{([1]_q - [2]_q)c^3}{2(\alpha)(2(\alpha))^2(\beta)(\delta)} - \frac{([1]_q - 2q)c^3}{(\delta)(2(\alpha))^3}.$$

As  $G(c) = \frac{(4 - c^2)c}{8(\delta)} + \frac{(4 - c^2)c}{8(\beta)(\alpha)} + \frac{(4 - c^2)c}{8(\delta)} + \frac{(4 - c^2)c([5]_q - [2]_q)}{8(\beta)(\alpha)(\delta)} - \frac{(4 - c^2)c}{4(\delta)} + \frac{(4 - c^2)}{4(\delta)} + \frac{c^3}{8(\beta)(\alpha)} + \frac{c^3}{8(\delta)} - \frac{(\beta)c^3}{8(\alpha)(\beta)(\delta)} - \frac{c^3}{8(\delta)} - \frac{c_3}{2(\delta)} + \frac{([1]_q - [2]_q)c^3}{2(\alpha)(2(\alpha))^2(\beta)} - \frac{c^3}{16(\beta)(\alpha)} - \frac{c^3}{48(\delta)} + \frac{([5]_q - [2]_q)c^3}{16(\alpha)(\delta)(\beta)} - \frac{(\alpha)c^3}{2(\alpha)(2(\alpha))^2(\beta)(\delta)} - \frac{([1]_q - 2q)c^3}{(\delta)(2(\alpha))^3}.$

Let  $G'(c) = 0$  as we get  $c = r = \frac{2(-2 + 4\sqrt{-q^2 + 3q})}{-q^2 + 5q + 3}$ .

We get G(c) put the values of c as

$$|a_2 a_3 - a_4| \leq 0.5351q. \tag{22}$$

Which are the required results.

**Theorem 7.** If  $f \in S_{q,l}^*$  then  $|a_2 a_4 - a_3^2| \leq \frac{4}{(2(\beta))^2} + \frac{1}{(\delta)(\alpha)}$ .

**Proof:** From (17) of theorem 5, we have

$$|a_2 a_4 - a_3^2| = \left| \left( \frac{c_1 c_3}{4(\delta)(\alpha)} - \frac{c_1^2 c_2}{8(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c_1^2 c_2}{8(\beta)(\alpha)^2(\delta)} + \frac{2c_2 c_1^2}{2(\beta)8(\beta)} - \frac{2c_2 c_1^2 [1]_q}{(2(\beta))^3(\beta)} + \frac{2c_2 c_1^2 [2]_q}{(2(\beta))^3(\beta)} + \frac{c_1^4}{96(\delta)(\alpha)} - \frac{c_1^4([5]_q - [2]_q)}{32(\beta)(\alpha)^2(\delta)} + \frac{c_1^4([5]_q - [2]_q)([1]_q - [2]_q)}{(\beta)(2(\alpha))^2(\delta)4(\alpha)^2} + \frac{c_1^4([1]_q - [2]_q)}{2(\alpha)(\delta)(2(\alpha))^3} \right) + \left( -\frac{c_2^2}{(2(\beta))^2} - \frac{c_1^4}{(8(\beta))^2} + \frac{2c_1^4 [1]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c_1^4 [2]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c_1^4 [1]^2 q}{(\beta)^2(2(\beta))^4} + \frac{2c_1^4 [1]_q [2]_q}{(\beta)^2(2(\beta))^4} - \frac{2c_1^4 [2]^2 q}{(\beta)^2(2(\beta))^4} \right) \right|.$$

We use the Lemma 2 we have

$$= \left| \frac{c_1^4}{16(\delta)(\alpha)} + \frac{c_1^2 x(4-c_1^2)}{8(\delta)(\alpha)} - \frac{c_1^2 x^2(4-c_1^2)}{16(\delta)(\alpha)} + \frac{c_1(4-c_1^2)(1-|x|^2 z)}{8(\delta)(\alpha)} - \frac{c_1^4}{16(\delta)(\alpha)} + \frac{c_1^2 x(4-c_1^2)}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c_1^4}{16(\beta)(\alpha)^2(\delta)} + \frac{([5]_q - [2]_q)c_1^2 x(4-c_1^2)}{16(\beta)(\alpha)^2(\delta)} + \frac{c_1^4}{2(\beta)8(\beta)} + \frac{c_1^2 x(4-c_1^2)}{2(\beta)8(\beta)} - \frac{[1]_q c_1^4}{(2(\beta))^3(\beta)} + \frac{c_1^2 [1]_q x(4-c_1^2)}{(2(\beta))^3(\beta)} + \frac{c_1^4}{(2(\beta))^3(\beta)} + \frac{c_1^2 [2]_q x(4-c_1^2)}{(2(\beta))^3(\beta)} + \frac{c_1^4}{96(\delta)(\alpha)} - \frac{c_1^4([5]_q - [2]_q)}{32(\beta)(\alpha)^2(\delta)} + \frac{c_1^4([5]_q - [2]_q)([1]_q - [2]_q)}{(\beta)(2(\alpha))^2(\delta)4(\alpha)^2} + \frac{c_1^4([1]_q - [2]_q)}{2(\alpha)(\delta)(2(\alpha))^3} - \frac{c_1^4}{4(2(\beta))^2} + \frac{c_1^2 x(4-c_1^2)}{2(2(\beta))^2} - \frac{x^2(4-c_1^2)^2}{4(2(\beta))^2} - \frac{c_1^4}{(8(\beta))^2} + \frac{2c_1^4 [1]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c_1^4 [2]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c_1^4 [1]^2 q}{(\beta)^2(2(\beta))^4} + \frac{2c_1^4 [1]_q [2]_q}{(\beta)^2(2(\beta))^4} - \frac{2c_1^4 [2]^2 q}{(\beta)^2(2(\beta))^4} \right|.$$

Denotes  $|x| = t \in [0,1], c_1 = c \in [0, 2]$  then using triangle inequality we have

$$|a_2 a_4 - a_3^2| \leq \frac{c^2 t(4-c^2)}{8(\delta)(\alpha)} + \frac{c^2 t(4-c^2)}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c^2 t(4-c^2)}{16(\beta)(\alpha)^2(\delta)} + \frac{c^2 t(4-c^2)}{2(\beta)8(\beta)} + \frac{c^2 [1]_q t(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 [1]_q t(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 [2]_q t(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 t(4-c^2)}{2(2(\beta))^2} + \frac{c^2 t^2(4-c^2)}{16(\delta)(\alpha)} + \frac{t^2(4-c^2)^2}{4(2(\beta))^2} + \frac{(4-c^2)}{4(\delta)(\alpha)} - \frac{c^4}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c^4}{16(\beta)(\alpha)^2(\delta)} + \frac{c^4}{16(\delta)(\alpha)} + \frac{c^4}{2(\beta)8(\beta)} - \frac{[1]_q c^4}{(2(\beta))^3(\beta)} + \frac{c^4}{(2(\beta))^3(\beta)} + \frac{c^4}{96(\delta)(\alpha)} - \frac{c^4([5]_q - [2]_q)}{32(\beta)(\alpha)^2(\delta)} + \frac{c^4([5]_q - [2]_q)([1]_q - [2]_q)}{(\beta)(2(\alpha))^2(\delta)4(\alpha)^2} + \frac{c^4([1]_q - [2]_q)}{2(\alpha)(\delta)(2(\alpha))^3} - \frac{c^4}{4(2(\beta))^2} - \frac{c^4}{(8(\beta))^2} + \frac{2c^4 [1]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c^4 [2]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c^4 [1]^2 q}{(\beta)^2(2(\beta))^4} + \frac{2c^4 [1]_q [2]_q}{(\beta)^2(2(\beta))^4} - \frac{2c^4 [2]^2 q}{(\beta)^2(2(\beta))^4}. \tag{23}$$

Which implies that

$$F(c, t) = \frac{c^2 t(4-c^2)}{8(\delta)(\alpha)} + \frac{c^2 t(4-c^2)}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c^2 t(4-c^2)}{16(\beta)(\alpha)^2(\delta)} + \frac{c^2 t(4-c^2)}{2(\beta)8(\beta)} + \frac{c^2 [1]_q t(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 [1]_q t(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 [2]_q t(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 t(4-c^2)}{2(2(\beta))^2} + \frac{c^2 t^2(4-c^2)}{16(\delta)(\alpha)} + \frac{t^2(4-c^2)^2}{4(2(\beta))^2} + \frac{(4-c^2)}{4(\delta)(\alpha)} - \frac{c^4}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c^4}{16(\beta)(\alpha)^2(\delta)} + \frac{c^4}{16(\delta)(\alpha)} + \frac{c^4}{2(\beta)8(\beta)} - \frac{[1]_q c^4}{(2(\beta))^3(\beta)} + \frac{c^4}{(2(\beta))^3(\beta)} + \frac{c^4}{96(\delta)(\alpha)} - \frac{c^4([5]_q - [2]_q)}{32(\beta)(\alpha)^2(\delta)} + \frac{c^4([5]_q - [2]_q)([1]_q - [2]_q)}{(\beta)(2(\alpha))^2(\delta)4(\alpha)^2} + \frac{c^4([1]_q - [2]_q)}{2(\alpha)(\delta)(2(\alpha))^3} - \frac{c^4}{4(2(\beta))^2} - \frac{c^4}{(8(\beta))^2} + \frac{2c^4 [1]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c^4 [2]_q}{8(\beta)(\beta)(2(\beta))^2} - \frac{2c^4 [1]^2 q}{(\beta)^2(2(\beta))^4} + \frac{2c^4 [1]_q [2]_q}{(\beta)^2(2(\beta))^4} - \frac{2c^4 [2]^2 q}{(\beta)^2(2(\beta))^4}. \tag{24}$$

We have

$$\frac{\partial F}{\partial t} = \frac{c^2(4-c^2)}{8(\delta)(\alpha)} + \frac{c^2(4-c^2)}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c^2(4-c^2)}{16(\beta)(\alpha)^2(\delta)} + \frac{c^2(4-c^2)}{16(\beta)^2} + \frac{c^2 [1]_q(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 [1]_q(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 [2]_q(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2(4-c^2)}{2(2(\beta))^2} + \frac{c^2 t(4-c^2)}{8(\delta)(\alpha)} + \frac{t(4-c^2)^2}{2(2(\beta))^2} \geq 0.$$

It demonstrates that F(c, t) has a maximum value at  $t = 1$  and that it is rising for every  $t$  in  $[0, 1]$ .

$$\begin{aligned}
 \text{Max } F(c, t) = F(c, t) &= \frac{c^2(4-c^2)}{8(\delta)(\alpha)} + \frac{c^2(4-c^2)}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c^2(4-c^2)}{16(\beta)(\alpha)^2(\delta)} + \frac{c^2(4-c^2)}{16(\beta)^2} + \frac{c^2 [1]_q(4-c^2)}{(2(\beta))^3(\beta)} + \\
 &\frac{c^2 [1]_q(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2 [2]_q(4-c^2)}{(2(\beta))^3(\beta)} + \frac{c^2(4-c^2)}{2(2(\beta))^2} + \frac{c^2(4-c^2)}{16(\delta)(\alpha)} + \frac{(4-c^2)^2}{4(2(\beta))^2} + \frac{(4-c^2)}{4(\delta)(\alpha)} - \frac{c^4}{16(\delta)(\alpha)} + \frac{([5]_q - [2]_q)c^4}{16(\beta)(\alpha)^2(\delta)} + \frac{c^4}{16(\delta)(\alpha)} + \frac{c^4}{16(\beta)^2} - \\
 &\frac{[1]_q c^4}{(2(\beta))^3(\beta)} + \frac{c^4}{(2(\beta))^3(\beta)} + \frac{c^4}{96(\delta)(\alpha)} - \frac{c^4([5]_q - [2]_q)}{32(\beta)(\alpha)^2(\delta)} + \frac{c^4([5]_q - [2]_q)([1]_q - [2]_q)}{(\beta)(2(\alpha))^2(\delta)4(\alpha)^2} + \frac{c^4([1]_q - [2]_q)}{2(\alpha)(\delta)(2(\alpha))^3} - \frac{c^4}{4(2(\beta))^2} - \frac{c^4}{(8(\beta))^2} + \\
 &\frac{2c^4 [1]_q}{8(\beta)^2(2(\beta))^2} - \frac{2c^4 [2]_q}{8(\beta)^2(2(\beta))^2} - \frac{2c^4 [1]_q^2}{(\beta)^2(2(\beta))^4} + \frac{2c^4 [1]_q[2]_q}{(\beta)^2(2(\beta))^4} - \frac{2c^4 [2]_q^2}{(\beta)^2(2(\beta))^4}.
 \end{aligned}
 \tag{25}$$

$M'(e)$  vanishes at  $e = 0$ , a straightforward computation yields  $M''(e) < 0$ , which implies that the function  $M(e)$  attains its maximum value at  $e=0$ . Hence, we obtain

$$|a_2 a_4 - a_3^2| \leq M(0) = \frac{4}{(2(\beta))^2} + \frac{1}{(\delta)(\alpha)}.
 \tag{26}$$

Thus, the desired results are obtained.

**Theorem 8.** If  $\in S_q^*$  then  $|H_3(1)| \leq \frac{1}{\beta} (0.2526861111 (\frac{3}{(2\beta)^2} + \frac{3}{4([4]_q(\alpha - [1]_q(\alpha))}))q$ .

**Proof:**

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2), a_1 = 1.
 \tag{27}$$

By applying triangle inequality we have

$$|H_3(1)| = |a_3||a_2 a_4 - a_3^2| + |a_4||a_4 - a_2 a_3| + |a_5||a_3 - a_2^2|.$$

Now, substituting the equation (19), (22), (26), and lemma 4 in (27) we get

$$|H_3(1)| = \frac{1}{\beta} (0.2526861111 (\frac{3}{(2\beta)^2} + \frac{3}{4([4]_q(\beta - [1]_q(\alpha))}))q.$$

**Discussion**

These constitute the required results. Researchers in the field of *Geometric Function Theory* have long been interested in studying the coefficients of univalent functions. In recent years, concepts from quantum theory and *q-calculus* have been incorporated into Geometric Function Theory to extend existing function classes and refine analytical results. Within this framework, the *Fekete–Szegő problem* has emerged as a classical and significant coefficient problem, with numerous applications in various branches of mathematical analysis. The *Hankel determinant* represents an important application of the Fekete–Szegő functional. While earlier studies primarily focused on the Hankel determinant of order two, more recent investigations have examined the third-order Hankel determinant for starlike functions with respect to symmetric points.

In this study, we investigate the third-order Hankel determinant for a newly introduced class of starlike functions defined with respect to the *q-exponential function* subordination involving symmetric points. The analysis utilizes the *q-derivative operator* and fundamental concepts from *quantum calculus*. Furthermore, it is noted that, by taking  $\rightarrow 1^-$ , the results obtained in this work reduce to those previously established in [36].

**Conclusion:**

In this article, we have introduced and analyzed a new class  $S_{q_l}^*$  of starlike functions with respect to symmetric points, associated with *q-exponential functions* through the subordination technique. Several significant results have been obtained, including coefficient inequalities, the *Fekete–Szegő problem*, and the third-order *Hankel determinant*. The findings demonstrate that the proposed class, along with the corresponding main theorems, extends and advances the results and function classes previously investigated by researchers in the field of Geometric Function Theory.

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